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On the best constants in inequalities of the Markov and Wirtinger types for polynomials on the half-line

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ABSTRACT

The main topic of the paper is best constants in Markov-type inequalities between the norms of higher derivatives of polynomials and the norms of the polynomials themselves. The norm is the L^2 norm with Laguerre weight. The leading term of the asymptotics of the constants is determined and tight bounds for the principal coefficient in this term, which is the operator norm of a Volterra operator, are given. For best constants in inequalities of the Wirtinger type, the limit is computed and an asymptotic formula for the error term is presented.

1. Introduction and main results

Let $\mathscr{P}_n$ denote the linear space of all complex polynomials of degree at most n and equip $\mathscr{P}_n$ with the norm

$$\|f\| = \left(\int_0^\infty |f(t)|^2 e^{-t}\,dt\right)^{1/2}. \tag{1}$$

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This paper addresses the best possible constants $\gamma_n^{(\nu)}$ for which

$$\|f^{(\nu)}\| \leqslant \gamma_n^{(\nu)}\|f\| \quad \text{for all } f \in \mathscr{P}_n. \tag{2}$$

Here $f^{(\nu)}$ stands for the νth derivative of f. Inequalities of this type go back to V.A. Markov [17]. In 1944, Erhard Schmidt [19] proved that

$$\gamma_n^{(1)} = \frac{2n+1}{\pi}\left(1 + \frac{\pi^2}{24(2n+1)^2} + O\left(\frac{1}{n^4}\right)\right), \tag{3}$$

which implies that $\gamma_n^{(1)} \sim (2/\pi)n$, where $x_n \sim y_n$ means that $x_n/y_n \to 1$ as $n \to \infty$. In 1960, Turán [24] found the exact value of $\gamma_n^{(1)}$:

$$\gamma_n^{(1)} = \left(2\sin\frac{\pi}{4n+2}\right)^{-1}.$$

Shortly after that, Shampine [20] showed that $\gamma_n^{(2)} \sim (1/\mu^2)n^2$ where μ is the smallest root of the equation $1 + \cos\mu\cosh\mu = 0$. In [7], it was observed that $\gamma_n^{(\nu)}$ is the largest singular value of some matrix A_n and this was used in [8] to establish the asymptotic estimates

$$\frac{1}{2\nu!}\sqrt{\frac{4}{2\nu+1}} \leqslant \liminf_{n\to\infty}\frac{\gamma_n^{(\nu)}}{n^\nu} \leqslant \limsup_{n\to\infty}\frac{\gamma_n^{(\nu)}}{n^\nu} \leqslant \frac{1}{2\nu!}\sqrt{\frac{2\nu}{2\nu-1}}. \tag{4}$$

It was not known whether $\gamma_n^{(\nu)}/n^\nu$ has a limit as $n \to \infty$ if $\nu \geqslant 3$. The following theorem shows that this limit exists and identifies it as the norm of a certain integral operator.

Theorem 1.1. *For every* $\nu \geqslant 1$,

$$\lim_{n\to\infty}\frac{\gamma_n^{(\nu)}}{n^\nu} = \|K_\nu\|_\infty$$

where $\|K_\nu\|_\infty$ *is the operator norm of the integral operator* K_ν *on* $L^2(0,1)$ *that is given by*

$$(K_\nu f)(x) = \frac{1}{(\nu-1)!}\int_x^1 (y-x)^{\nu-1}f(y)\,dy. \tag{5}$$

Moreover, $\gamma_n^{(\nu)} = \|K_\nu\|_\infty n^\nu + O(n^{\nu-1})$.

We are thus led to the problem of finding the norm $\|K_\nu\|_\infty$. This problem, which has been made popular by Paul Halmos, was studied by many authors. It is well-known since [12] that $\|K_1\|_\infty = 2/\pi$. Thorpe [23] showed that $\|K_\nu\|_\infty = 1/\mu^\nu$ where μ is the smallest positive number such that the equation $(-1)^\nu g^{(2\nu)} = \mu^{2\nu}g$ with the boundary conditions $g^{(j)}(0) = 0$ for $j = 0,\dots,\nu-1$ and $g^{(j)}(1) = 0$ for $j = \nu,\dots,2\nu-1$ has a nontrivial solution. For $\nu = 1,2,3$, this gives that μ is the smallest positive number satisfying $\cos\mu = 0$, $\cos\mu = -1/\cosh\mu$, and

$$\cos\mu = -\frac{1}{\cosh\mu\sqrt{3}}\left[8\cosh\frac{\mu\sqrt{3}}{2}\cos\frac{\mu}{2} + (2+\cos\mu)^2\right],$$

respectively; see [5] for a way of getting the last equation. For $1 \leqslant \nu \leqslant 25$, numerical values for μ and thus for $\|K_\nu\|_\infty$ are in [11]. Thorpe [23] also proved that $\|K_\nu\|_\infty \sim 1/(2\nu!)$ as $\nu \to \infty$. Independently, Lao and Whitley [15] computed the norms for $1 \leqslant \nu \leqslant 10$ numerically and conjectured that $\|K_\nu\|_\infty \sim 1/(2\nu!)$. This conjecture, which was already proved at that time by Thorpe, was then independently also proved in [14,16]. Further generalizations are in [1,10]. We here prove the following.

Theorem 1.2. *We have*

$$\frac{1}{(\nu-1)!}\frac{1}{\sqrt{(2\nu+1)(2\nu-1)}} \leqslant \|K_\nu\|_\infty \leqslant \frac{1}{(\nu-1)!}\frac{1}{\sqrt{2\nu(2\nu-1)}} \tag{6}$$

for all $\nu \geqslant 1$.

The upper bound in (6) is simply the Hilbert–Schmidt norm $\|K_\nu\|_2$ of the operator K_ν and known since [15]. The lower bound is better than any previous lower bound we are aware of, in particular sharper than the lower bounds

$$\frac{1}{(\nu-1)!}\frac{1}{\sqrt{2\nu-1}(2\nu)^{\nu/(2\nu-1)}}, \quad \frac{1}{(\nu-1)!}\frac{1}{2\nu}\left(1+\frac{1}{2\nu}\right)^{-1/2}, \quad \frac{1}{(\nu-1)!}\frac{1}{2\nu},$$

which were established in [23,1,14], respectively. Moreover, our proof of the lower bound in (6) is extremely short and lucid. It is based on an estimation of three lines that gives a lower bound for $\|K_\nu^* K_\nu\|_2/\|K_\nu\|_2$, which is in turn a lower bound for $\|K_\nu\|_\infty$. Note that (6) immediately implies that

$$\|K_\nu\|_\infty = \frac{1}{2\nu!}\left(1+O\left(\frac{1}{\nu}\right)\right).$$

We finally derive bounds for $\|K_\nu\|_\infty$ that are even tighter than those in (6). Let

$$I(\nu) = 2\int_0^1\int_0^x\left(\int_0^y (x-t)^{\nu-1}(y-t)^{\nu-1}\,dt\right)^2 dy\,dx. \tag{7}$$

An alternative expression for $I(\nu)$ is

$$I(\nu) = \frac{1}{2\nu^3}\left[\sum_{k=1}^{\nu}\frac{1}{2\nu-1+2k}\frac{(-\nu)_k^2}{(\nu)_k^2} + 2\sum_{k=2}^{\nu}\sum_{j=1}^{k-1}\frac{1}{2\nu-1+k+j}\frac{(-\nu)_k}{(\nu)_k}\frac{(-\nu)_j}{(\nu)_j}\right], \tag{8}$$

where $(z)_n := z(z+1)\dots(z+n-1)$.

Theorem 1.3. *For every $\nu \geqslant 1$,*

$$\frac{1}{2(\nu-1)!\sqrt{\nu(2\nu-1)}}\sqrt{1+\sqrt{8\nu^2(2\nu-1)^2 I(\nu)-1}} \leqslant \|K_\nu\|_\infty \leqslant \frac{\sqrt[4]{I(\nu)}}{(\nu-1)!}. \tag{9}$$

We will show that the lower bound in (9) is better than those in (4) and (6) and that the upper bound in (9) is at least as good as those in (4) and (6). Bounds (9) rapidly become sharper and sharper as ν increases. They deliver the values for $\|K_\nu\|_\infty$ shown in the table. The indicated digits of these values are correct because the bounds coincide in these digits.

	$\|K_\nu\|_\infty$
$\nu = 1$	$0.6\dots$
$\nu = 2$	$0.284\dots$
$\nu = 3$	$0.09081\dots$
$\nu = 4$	$0.022213\dots$
$\nu = 5$	$0.0043851\dots$
$\nu = 6$	$7.2456\dots \times 10^{-4}$
$\nu = 7$	$1.02874\dots \times 10^{-4}$
$\nu = 8$	$1.28003\dots \times 10^{-5}$
$\nu = 9$	$1.417196\dots \times 10^{-6}$
$\nu = 10$	$1.413169\dots \times 10^{-7}$
$\nu = 20$	$2.0811690\dots \times 10^{-19}$
$\nu = 40$	$6.16662705\dots \times 10^{-49}$
$\nu = 60$	$6.034043870\dots \times 10^{-83}$

The reverse of inequality (2) reads $\|f\| \leqslant \beta_n^{(\nu)}\|f^{(\nu)}\|$, and since $f^{(\nu)} = 0$ for $f \in \mathscr{P}_{\nu-1}$, there is clearly no finite $\beta_n^{(\nu)}$ such that this inequality holds for all $f \in \mathscr{P}_n$. However, there exist $\beta_n^{(\nu)} < \infty$ such that

$$\|f\| \leqslant \beta_n^{(\nu)}\|f^{(\nu)}\| \quad \text{for all } f \in \mathscr{P}_n \ominus \mathscr{P}_{\nu-1}, \tag{10}$$

where $\mathscr{P}_n \ominus \mathscr{P}_{\nu-1}$ is the orthogonal complement of $\mathscr{P}_{\nu-1}$ in $\mathscr{P}_n$, that is, the linear span of the Laguerre polynomials $L_\nu, L_{\nu+1}, \dots, L_n$. Inequalities of the type (10) are usually referred to as Wirtinger or Wirtinger–Sobolev inequalities. See [18] for more on this subject, including a reliable history, and [6] for the connection with Toeplitz matrices. We here prove the following.

Theorem 1.4. *For $\nu = 1$ and $n \geqslant 1$ the best possible $\beta_n^{(\nu)}$ in (10) is*

$$\beta_n^{(1)} = 2\cos\frac{\pi}{2n+1}, \tag{11}$$

and for each $\nu \geqslant 1$ the best possible $\beta_n^{(\nu)}$ in (10) satisfy

$$\beta_n^{(\nu)} = 2^\nu\left(1 - \frac{\pi^2\nu}{8n^2}\right) + O\left(\frac{1}{n^3}\right) \tag{12}$$

as $n \to \infty$.

The paper is organized as follows. In Section 2, we introduce a triangular Toeplitz matrix A_n such that the best possible constants in (2) and (10) are just the spectral norms of A_n and A_n^{-1}, respectively. In Section 3 we prove Theorem 1.1 and Section 4 is devoted to the proof of Theorems 1.2 and 1.3. The techniques employed in Section 3 work for the norm of A_n but not for the norm of A_n^{-1}. Fortunately, we may have almost immediate recourse to known results to tackle the norm of A_n^{-1} and thus to prove Theorem 1.4. This is the subject of Section 5.

2. Matrix representation of the operator of differentiation

In this section we follow [7–9]. The Laguerre polynomials are given by

$$L_k(t) = 1 - \binom{k}{1}\frac{t}{1!} + \binom{k}{2}\frac{t^2}{2!} - \cdots + (-1)^k\binom{k}{k}\frac{t^k}{k!},$$

and $\mathscr{E} = \{L_0, L_1, \ldots, L_n\}$ is an orthonormal basis in $\mathscr{P}_n$ with the norm (1). We have

$$L_k' = -L_0 - L_1 - \cdots - L_{k-1}$$

and hence the matrix representation of the operator $f \mapsto f'$ on $\mathscr{P}_n$ in $\mathscr{E}$ is the $(n+1) \times (n+1)$ triangular Toeplitz matrix

$$D = \begin{pmatrix} 0 & -1 & -1 & \ldots & -1 \\ & 0 & -1 & \ldots & -1 \\ & & & & \ldots \\ & & & & -1 \\ & & & & 0 \end{pmatrix}.$$

This implies that the operator $f \mapsto f^{(\nu)}$ is represented in $\mathscr{E}$ by the $(n+1) \times (n+1)$ triangular Toeplitz matrix

$$D^\nu = (-1)^\nu \begin{pmatrix} 0 & \binom{0}{\nu-1} & \binom{1}{\nu-1} & \ldots & \binom{n-1}{\nu-1} \\ & 0 & \binom{0}{\nu-1} & \ldots & \binom{n-2}{\nu-1} \\ & & & & \ldots \\ & & & & \binom{0}{\nu-1} \\ & & & & 0 \end{pmatrix}.$$

The best possible constant $\gamma_n^{(\nu)}$ in (2) is therefore nothing but the norm of the operator given by the matrix D^ν on $\mathbf{C}^{n+1}$ with the ℓ^2 norm. Since

$$D^\nu = (-1)^\nu \begin{pmatrix} 0 & A_n \\ 0 & 0 \end{pmatrix}$$

with the $(n-\nu+1)\times(n-\nu+1)$ triangular Toeplitz matrix

$$A_n = \begin{pmatrix} \binom{\nu-1}{\nu-1} & \binom{\nu}{\nu-1} & \dots & \binom{n-1}{\nu-1} \\ & \binom{\nu-1}{\nu-1} & \dots & \binom{n-2}{\nu-1} \\ & & & \dots \\ & & & \binom{\nu-1}{\nu-1} \end{pmatrix}, \tag{13}$$

$\gamma_n^{(\nu)}$ is the operator norm of A_n. (The dependence of A_n on ν will be suppressed.) Note that

$$A_n = (a_{j-k})_{j,k=0}^{n-\nu} \quad \text{with} \quad a_{-\ell} = \binom{\nu-1+\ell}{\nu-1} \tag{14}$$

for $\ell \geqslant 0$ and $a_\ell = 0$ for $\ell \geqslant 1$.

Given a matrix $A \in \mathbf{C}^{N\times N}$, we denote by $\|A\|_\infty$ the operator norm (=spectral norm = largest singular value) and by $\|A\|_2$ the Hilbert–Schmidt norm (=Frobenius norm) of A. Recall that if $A = (a_{jk})$, then $\|A\|_2^2 = \sum_{j,k} |a_{jk}|^2$.

An upper bound for $\|A_n\|_\infty$ follows from the trivial estimate

$$\|A_n\|_\infty^2 \leqslant \|A_n^* A_n\|_2,$$

where A^* is the adjoint matrix of A (=transpose of A in the case of real matrices). Things are a little more involved when looking for lower bounds. Using a result by Tarazaga [21,22], it was observed in [9] that if A is a matrix in $\mathbf{R}^{N\times N}$ such that

$$\text{rank} A \geqslant 2 \quad \text{and} \quad \frac{\|A\|_2^4}{\|A^*A\|_2^2} \leqslant 2, \tag{15}$$

then

$$\|A\|_\infty^2 \geqslant \frac{\|A\|_2^2}{2} + \left[\frac{1}{2}\left(\|A^*A\|_2^2 - \frac{\|A\|_2^4}{2}\right)\right]^{1/2}. \tag{16}$$

We will see that this estimate delivers very good results for the matrices (13).

3. The asymptotics of the norms

In [25–27], Harold Widom introduced a great idea in order to find the asymptotics of norms of certain matrices as their dimension goes to infinity: he replaced the matrix $(a_{jk})_{j,k=0}^{N-1}$ by the integral operator on $L^2(0,1)$ with the piecewise constant kernel $g_N(x,y) = a_{[Nx],[Ny]}$, where $[\xi]$ is the integral part of ξ, and analyzed whether this integral operator is connected with a limiting integral operator with some nice kernel $k(x,y)$. We here proceed in the same way. See also [4], where this idea was employed in a similar but more complicated situation. Interestingly, Lao and Whitley [15] proceeded in the reverse direction: they approximated Volterra integral operators by integral operators with piecewise constant kernels and thus by matrices and then numerically computed the norm of the matrices in order to get approximations for the norm of the integral operators.

Let K be the integral operator on $L^2(0,1)$ given by

$$(Kf)(x) = \int_0^1 k(x,y)f(y)\,dy.$$

The Hilbert–Schmidt norm of this operator is defined by

$$\|K\|_2 = \left(\int_0^1 \int_0^1 |k(x,y)|^2\,dx\,dy\right)^{1/2}.$$

If $\|K\|_2$ is finite, then K is bounded on $L^2(0,1)$ and for the operator norm $\|K\|_\infty$ of K the estimate $\|K\|_\infty \leqslant \|K\|_2$ holds. The adjoint operator is given by

$$(K^*f)(x) = \int_0^1 \overline{k(y,x)} f(y)\, dy.$$

Lemma 3.1 (**Widom**). *Let $A_N = (a_{jk})_{j,k=0}^{N-1}$ be a matrix in $\mathbf{C}^{N\times N}$ and let G_N be the integral operator on $L^2(0,1)$ with the kernel*

$$g_N(x,y) = a_{[Nx],[Ny]}, \quad (x,y) \in (0,1)^2.$$

Then

$$\|A_N\|_\infty = N\|G_N\|_\infty, \quad \|A_N\|_2 = N\|G_N\|_2, \quad \|A_N^*A_N\|_2 = N^2\|G_N^*G_N\|_2.$$

Proof. Let I_k be the interval $(k/N,(k+1)/N)$, denote by χ_{I_k} the characteristic function of I_k, and consider the operators

$$S_N : \mathbf{C}^N \to L^2(0,1), \quad \{x_k\}_{k=0}^{N-1} \mapsto \sqrt{N}\sum_{k=0}^{N-1} x_k\chi_{I_k},$$

$$R_N : L^2(0,1) \to \mathbf{C}^N, \quad f \mapsto \left\{\sqrt{N}\int_{I_k} f(x)\, dx\right\}_{k=0}^{N-1}.$$

It is easily seen that $\|S_N\|_\infty = \|R_N\|_\infty = 1$, that R_NS_N is the identity operator on $\mathbf{C}^N$, and that $S_NA_NR_N = NG_N$ and $A_N = NR_NG_NS_N$. Thus,

$$\begin{aligned}
N\|G_N\|_\infty &\leqslant \|S_N\|_\infty\|A_N\|_\infty\|R_N\|_\infty = \|A_N\|_\infty \\
&\leqslant N\|R_N\|_\infty\|G_N\|_\infty\|S_N\|_\infty = N\|G_N\|_\infty, \\
N\|G_N\|_2 &\leqslant \|S_N\|_\infty\|A_N\|_2\|R_N\|_\infty = \|A_N\|_2 \\
&\leqslant N\|R_N\|_\infty\|G_N\|_2\|S_N\|_\infty = N\|G_N\|_2,
\end{aligned}$$

which proves the first two asserted equalities. Since $S_NA_N^*R_N = NG_N^*$, we get

$$S_NA_N^*A_NR_N = S_NA_N^*R_NS_NA_NR_N = N^2G_N^*G_N$$

and thus $A_N^*A_N = N^2R_NG_N^*G_NS_N$. This yields the third of the asserted equalities as above. □

Now let K_ν be the integral operator (5). Note that

$$(K_\nu^*f)(x) = \frac{1}{(\nu-1)!}\int_0^x (x-y)^{\nu-1}f(y)\, dy. \tag{17}$$

Theorem 3.2. *Let the matrices A_n be given by (13) and (14). If $\nu = 1$, then*

$$\|A_n\|_\infty = \|K_1\|_\infty n + O(1), \quad \|A_n\|_2 = \|K_1\|_2 n\left(1+\frac{1}{n}\right)^{1/2}, \tag{18}$$

$$\|A_n^*A_n\|_2 = \|K_1^*K_1\|_2 n^2 + O(n^{3/2}), \tag{19}$$

and if $\nu \geqslant 2$, then

$$\|A_n\|_\infty = \|K_\nu\|_\infty n^\nu + O(n^{\nu-1}), \quad \|A_n\|_2 = \|K_\nu\|_2 n^\nu + O(n^{\nu-1}), \tag{20}$$

$$\|A_n^*A_n\|_2 = \|K_\nu^*K_\nu\|_2 n^{2\nu} + O(n^{2\nu-1}). \tag{21}$$

Proof. We use Lemma 3.1 with $N = n-\nu+1$ and A_n in place of A_N. The integral operator G_N has the kernel $a_{[Nx]-[Ny]}$. Let $K_{N,\nu} = (1/N^{\nu-1})G_N$. The kernel of $K_{N,\nu}$ is $k_{N,\nu}(x,y) = (1/N^{\nu-1})a_{[Nx]-[Ny]}$. Let $k_\nu(x,y)$ denote the kernel of K_ν. We can divide $[0,1]^2$ into N^2 squares of side-length $1/N$ so that the kernel $k_{N,\nu}$ is constant on each of these squares. Exactly N of the squares are centered on the diagonal $y = x$, half of the rest of the squares lie above this diagonal, and the other half are below the diagonal.

Suppose first that $\nu = 1$. Then both $k_{N,\nu}$ and k_ν are 1 on the squares above the diagonal and 0 on the squares below it. On the squares along the diagonal, $k_{N,\nu}$ is 1 while $k_\nu(x,y) = 1$ for $y > x$ and $k_\nu(x,y) = 0$ for $y < x$. Thus,

$$\|K_{N,\nu} - K_\nu\|_2^2 = \int_0^1 \int_0^1 |k_{N,\nu}(x,y) - k_\nu(x,y)|^2 \, dx \, dy = N \cdot \frac{1}{2N^2} = \frac{1}{2N},$$

$$\|K_{N,\nu}\|_2^2 = \int_0^1 \int_0^1 |k_{N,\nu}(x,y)|^2 \, dx \, dy = \frac{N(N+1)}{2} \cdot \frac{1}{N^2} = \frac{1}{2}\left(1 + \frac{1}{N}\right),$$

$$\|K_\nu\|_2^2 = \int_0^1 \int_0^1 |k_\nu(x,y)|^2 \, dx \, dy = \frac{1}{2}.$$

Since $N = n$ for $\nu = 1$, we deduce from Lemma 3.1 that

$$\|A_n\|_2 = n\|G_n\|_2 = n\|K_{n,\nu}\|_2 = \|K_\nu\|_2 n\sqrt{1 + 1/n},$$

which is the second formula in (18). Using that

$$\begin{aligned} &\|K_{N,\nu}^* K_{N,\nu} - K_\nu^* K_\nu\|_2 \\ &\quad \leqslant \|K_{N,\nu}^* - K_\nu^*\|_2 \|K_{N,\nu}\|_2 + \|K_\nu^*\|_2 \|K_{N,\nu} - K_\nu\|_2 = O\left(\frac{1}{\sqrt{N}}\right), \end{aligned} \tag{22}$$

we obtain that $\|K_{N,\nu}^* K_{N,\nu}\|_2 = \|K_\nu^* K_\nu\|_2 + O(1/\sqrt{N})$. As, again by Lemma 3.1 and the equality $N = n$,

$$\|A_n^* A_n\|_2 = n^2 \|G_n^* G_n\|_2 = n^2 \|K_{n,\nu}^* K_{n,\nu}\|_2,$$

it results that $\|A_n^* A_n\|_2 = \|K_\nu^* K_\nu\|_2 n^2 + O(n^{3/2})$, which is (19). The estimate $\|K_{N,\nu} - K_\nu\|_\infty \leqslant \|K_{N,\nu} - K_\nu\|_2$ gives the first formula of (18) with $O(\sqrt{n})$ instead of $O(1)$. Comparing this with Schmidt's asymptotics (3) we see that $\|K_1\|_\infty = 2/\pi$ and that the $O(\sqrt{n})$ is actually $O(1)$. This completes the proof in the case $\nu = 1$.

Now let $\nu \geqslant 2$. In that case

$$k_{N,\nu}(x,y) = \frac{1}{N^{\nu-1}} a_{[Nx]-[Ny]} = \frac{1}{N^{\nu-1}} \binom{\nu - 1 + [Ny] - [Nx]}{\nu - 1}$$

on the squares above the diagonal $y = x$. Writing $[Ny] - [Nx]$ in the form $N(y - x) + \delta_N(x,y)$ with $|\delta_N(x,y)| \leqslant 2$, we obtain that

$$\begin{aligned} k_{N,\nu}(x,y) &= \frac{1}{(\nu-1)!} \prod_{\ell=0}^{\nu-2} \frac{\nu - 1 + [Ny] - [Nx] - \ell}{N} \\ &= \frac{1}{(\nu-1)!} \prod_{\ell=0}^{\nu-2} \left(y - x + \frac{\nu - 1 + \delta_N(x,y) - \ell}{N}\right) \\ &= \frac{1}{(\nu-1)!} \prod_{\ell=0}^{\nu-2} (y - x) + O\left(\frac{1}{N}\right) = k_\nu(x,y) + O\left(\frac{1}{N}\right), \end{aligned}$$

the $O(1/N)$ uniformly in x and y. On the squares below the diagonal we have $k_{N,\nu}(x,y) = k_\nu(x,y) = 0$ and on the N squares along the diagonal, $k_{N,\nu}(x,y) = 1/N^{\nu-1}$ and $k_\nu(x,y) = O(1/N^{\nu-1})$ uniformly in x and y. Consequently,

$$\begin{aligned} \|K_{N,\nu} - K_\nu\|_2^2 &= \int_0^1 \int_0^1 |k_{N,\nu}(x,y) - k_\nu(x,y)|^2 \, dx \, dy \\ &= O\left(\frac{1}{N^2}\right) + N \cdot O\left(\frac{1}{N^{2\nu-2}}\right) \frac{1}{N^2} = O\left(\frac{1}{N^2}\right). \end{aligned}$$

This implies that $\|K_{N,\nu} - K_\nu\|_\infty \leqslant \|K_{N,\nu} - K_\nu\|_2 = O(1/N)$. It follows that

$$\|K_{N,\nu}\|_\infty = \|K_\nu\|_\infty + O\left(\frac{1}{N}\right), \quad \|K_{N,\nu}\|_2 = \|K_\nu\|_2 + O\left(\frac{1}{N}\right),$$

and since, by Lemma 3.1,

$$\|A_n\|_\infty = N\|G_N\|_\infty = N^\nu\|K_{N,\nu}\|_\infty, \quad \|A_n\|_2 = N\|G_N\|_2 = N^\nu\|K_{N,\nu}\|_2,$$

we arrive at the conclusion that

$$\|A_n\|_\infty = \|K_\nu\|_\infty N^\nu + O(N^{\nu-1}) = \|K_\nu\|_\infty n^\nu + O(n^{\nu-1}),$$
$$\|A_n\|_2 = \|K_\nu\|_2 N^\nu + O(N^{\nu-1}) = \|K_\nu\|_2 n^\nu + O(n^{\nu-1}),$$

which proves (20). Finally, we have again (22), but this time with $O(1/N)$ instead of $O(1/\sqrt{N})$. Thus, $\|K_{N,\nu}^*K_{N,\nu}\|_2 = \|K_\nu^*K_\nu\|_2 + O(1/N)$. From Lemma 3.1 we now infer that

$$\|A_n^*A_n\|_2 = N^2\|G_N^*G_N\|_2 = N^{2\nu}\|K_{N,\nu}^*K_{N,\nu}\|_2,$$

and hence

$$\|A_n^*A_n\|_2 = \|K_\nu^*K_\nu\|_2 N^{2\nu} + O(N^{2\nu-1}) = \|K_\nu^*K_\nu\|_2 n^{2\nu} + O(n^{2\nu-1}),$$

which is (21). $\square$

At this point we have proved Theorem 1.1, which is equivalent to the statement that $\|A_n\|_\infty = \|K_\nu\|_\infty n^\nu + O(n^{\nu-1})$.

4. Norms of Volterra operators

In this section we prove Theorems 1.2 and 1.3.

Lemma 4.1. *We have*

$$\|K_\nu\|_2^2 = \frac{1}{2\nu(2\nu-1)}\frac{1}{[(\nu-1)!]^2}, \quad \|K_\nu^*K_\nu\|_2^2 = \frac{I(\nu)}{[(\nu-1)!]^4}$$

for every $\nu \geqslant 1$.

Proof. From (5) we infer that

$$[(\nu-1)!]^2\|K_\nu\|_2^2 = \int_0^1\int_x^1 (y-x)^{2\nu-2}\,dy\,dx = \frac{1}{2\nu(2\nu-1)}.$$

Combining (5) and (17) we see that $K_\nu^*K_\nu$ is the integral operator with the kernel

$$c(x,y) = \frac{1}{[(\nu-1)!]^2}\int_0^{\min(x,y)} (x-t)^{\nu-1}(y-t)^{\nu-1}\,dt.$$

Since $c(x,y) = c(y,x)$, we get

$$\|K_\nu^*K_\nu\|_2^2 = 2\int_0^1\int_0^x c^2(x,y)dydx = \frac{I(\nu)}{[(\nu-1)!]^4}. \quad \square$$

Formula (8) can be derived from the integral representation (7) as follows. Substituting $t = \tau y$ in the inner integral, we see that this integral is

$$y^\nu \int_0^1 (x-\tau y)^{\nu-1}(1-\tau)^{\nu-1}\,d\tau$$
$$= y^\nu \sum_{k=1}^{\nu} \binom{\nu-1}{k-1} x^{\nu-k}(-1)^{k-1}y^{k-1} \int_0^1 \tau^{k-1}(1-\tau)^{\nu-1}\,d\tau$$
$$= \sum_{k=1}^{\nu} C(k,\nu)(-1)^{k-1}x^{\nu-k}y^{\nu+k-1} \tag{23}$$

with

$$C(k,\nu) = \binom{\nu-1}{k-1}\frac{\Gamma(k)\Gamma(\nu)}{\Gamma(k+\nu)} = \frac{(-1)^k}{\nu}\frac{(-\nu)_k}{(\nu)_k}.$$

The square of (23) is

$$\sum_{k=1}^{\nu} C(k,\nu)^2 x^{2\nu-2k}y^{2\nu+2k-2} + 2\sum_{k=2}^{\nu}\sum_{j=1}^{k-1} C(k,\nu)C(j,\nu)(-1)^{j+k}x^{2\nu-j-k}y^{2\nu+j+k-2}$$

and after integrating this over y from 0 to x and then over x from 0 to 1 we arrive exactly at (8).

Lemma 4.2. *For every $\nu \geqslant 1$,*

$$\frac{1}{2\nu(2\nu+1)(2\nu-1)^2} \leqslant I(\nu) \leqslant \frac{1}{4\nu^2(2\nu-1)^2}.$$

Proof. Lemma 4.1 implies that

$$I(\nu) = [(\nu-1)!]^4\|K_\nu^* K_\nu\|_2^2 \leqslant [(\nu-1)!]^4\|K_\nu\|_2^4 = \frac{1}{4\nu^2(2\nu-1)^2}.$$

Substituting $t = ys$ in (7) we get

$$I(\nu) = 2\int_0^1\int_0^x \left(\int_0^1 (x-ys)^{\nu-1}y^\nu(1-s)^{\nu-1}\,ds\right)^2 dy\,dx \tag{24}$$

and since $x - ys \geqslant x - xs$ for $0 \leqslant s$ and $0 \leqslant y \leqslant x$, it follows that

$$I(\nu) \geqslant 2\int_0^1\int_0^x \left(\int_0^1 x^{\nu-1}y^\nu(1-s)^{2\nu-2}ds\right)^2 dy\,dx$$
$$= 2\int_0^1\int_0^x \frac{x^{2\nu-2}y^{2\nu}}{(2\nu-1)^2}\,dy\,dx = \frac{2}{4\nu(2\nu+1)(2\nu-1)^2}. \qquad \square$$

We are now in a position to prove Theorem 1.2. The upper bound in (6) follows from Lemma 4.1 and the inequality $\|K_\nu\|_\infty \leqslant \|K_\nu\|_2$. On the other hand, from Lemmas 4.1 and 4.2 we obtain that

$$\|K_\nu\|_\infty^2 \geqslant \frac{\|K_\nu^* K_\nu\|_2^2}{\|K_\nu\|_2^2} = \frac{2\nu(2\nu-1)I(\nu)}{[(\nu-1)!]^2} \geqslant \frac{1}{(2\nu+1)(2\nu-1)}\frac{1}{[(\nu-1)!]^2},$$

which is the lower bound in (6).

Here is the proof of Theorem 1.3. The upper bound in (9) is immediate from Lemma 4.1 and the estimate $\|K_\nu\|_\infty^2 = \|K_\nu^* K_\nu\|_\infty \leqslant \|K_\nu^* K_\nu\|_2$. To prove the lower bound, we first apply (16) to the matrices A_n given by (13). In order to employ (16) we have to guarantee hypothesis (15). It is clear that $\operatorname{rank} A_n = n - \nu + 1 \geqslant 2$ for $n \geqslant \nu + 1$. From Theorem 3.2 and Lemma 4.1 we know that

$$\frac{\|A_n\|_2^4}{\|A_n^* A_n\|_2^2} \to \frac{\|K_\nu\|_2^4}{\|K_\nu^* K_\nu\|_2^2} = \frac{1}{4\nu^2(2\nu-1)^2 I(\nu)}$$

and since, by Lemma 4.2,

$$\frac{1}{4\nu^2(2\nu-1)^2 I(\nu)} \leqslant \frac{2\nu(2\nu+1)(2\nu-1)^2}{4\nu^2(2\nu-1)^2} = \frac{2\nu+1}{2\nu} < 2$$

for $\nu \geqslant 1$, there exists an $n_0(\nu)$ such that $\|A_n\|_2^4/\|A_n^* A_n\|_2^2 < 2$ for all $n \geqslant n_0(\nu)$. Thus, we may indeed apply (16) to the matrices A_n. Inserting A_n for A in (16), dividing the result by $N^{2\nu} = (n-\nu+1)^{2\nu}$, and finally taking into account Theorem 3.2 and Lemma 4.1, we get

$$\begin{aligned}\|K_\nu\|_\infty^2 &\geqslant \frac{\|K_\nu\|_2^2}{2} + \left[\frac{1}{2}\left(\|K_\nu^* K_\nu\|_2^2 - \frac{\|K_\nu\|_2^4}{2}\right)\right]^{1/2} \\ &= \frac{1}{4\nu(2\nu-1)[(\nu-1)!]^2}\left(1+\sqrt{8\nu^2(2\nu-1)^2 I(\nu)-1}\right),\end{aligned}$$

which completes the proof of Theorem 1.3.

We are left with the comparison of the bounds in (4), (6), (9). From Lemma 4.2 we infer that the upper bound in (9) does not exceed the upper bound in (6), and it is readily seen that the upper bounds in (6) and (4) coincide. Again by Lemma 4.2,

$$8\nu^2(2\nu-1)^2 I(\nu) - 1 \geqslant \frac{4\nu}{2\nu+1} - 1 = \frac{2\nu-1}{2\nu+1} > \left(\frac{2\nu-1}{2\nu+1}\right)^2,$$

whence

$$\frac{1+\sqrt{8\nu^2(2\nu-1)^2 I(\nu)-1}}{4\nu(2\nu-1)} > \frac{1+(2\nu-1)/(2\nu+1)}{4\nu(2\nu-1)} = \frac{1}{(2\nu+1)(2\nu-1)}.$$

This reveals that the lower bound in (9) is better than that in (6). Finally, it is elementary to see that the lower bound in (6) is equal to the lower bound in (4) for $\nu = 1$ and strictly larger than the lower bound in (4) for $\nu \geqslant 2$.

5. Inequalities of the Wirtinger type

We finally prove Theorem 1.4. We know from Section 2 that the operator $f \mapsto f^{(\nu)}$ is represented by the matrix D^ν in the basis $\mathscr{E}$. Taking into account that $\mathscr{P}_n \ominus \mathscr{P}_{\nu-1}$ is the linear space spanned by the $n-\nu+1$ Laguerre polynomials $L_\nu, \ldots, L_n$, we see that the operator

$$\mathscr{P}_n \ominus \mathscr{P}_{\nu-1} \to \mathscr{P}_{n-\nu}, \quad f \mapsto f^{(\nu)}$$

is unitarily equivalent to the operator induced by the matrix A_n on $\mathbf{C}^{n-\nu+1}$ with the ℓ^2 norm. Thus, the best constant $\beta_n^{(\nu)}$ in (10) is just $\|A_n^{-1}\|_\infty$. Put $N = n-\nu+1$. It is easily seen that A_n^{-1} is the $N \times N$ upper-triangular banded Toeplitz matrix whose first row is

$$\left[(-1)^0\binom{\nu}{0}, (-1)^1\binom{\nu}{1}, \ldots, (-1)^\nu\binom{\nu}{\nu}, 0, \ldots, 0\right].$$

In contrast to the sequence $\{A_n\}$, the sequence $\{A_n^{-1}\}$ is a sequence of principal truncations of an infinite Toeplitz matrix that is generated by the Fourier coefficients of an L^1 function. We use the common notation, such as in [2] or [3]. Given a function $a \in L^1(-\pi,\pi)$ with Fourier coefficients

$$a_\ell = \frac{1}{2\pi}\int_{-\pi}^{\pi} a(x)e^{-i\ell x}\,dx \quad (\ell \in \mathbf{Z}),$$

we let $T_N(a)$ stand for the $N \times N$ Toeplitz matrix $(a_{j-k})_{j,k=0}^{N-1}$. We then may write $A_n^{-1} = T_N(b)$ with $b(x) = (1-e^{-ix})^\nu$. In what follows it will be a little more convenient to work with $c(x) = (1+e^{-ix})^\nu$. It is easily seen that $T_N(c) = \Lambda T_N(b)\Lambda$ with $\Lambda = \mathrm{diag}(1,-1,1,-1,\ldots)$, so that

$$\|A_n^{-1}\|_\infty = \|T_N(b)\|_\infty = \|T_N(c)\|_\infty.$$

It is well known that if a is any function in $L^\infty(0,2\pi)$, then $\|T_N(a)\|_\infty$ converges monotonically to

$$\|a\|_\infty := \sup_{x\in(0,2\pi)} |a(x)|,$$

the supremum understood as the essential supremum. Since $\|c\|_\infty = 2^\nu$, we can at this point already say that $\|T_N(c)\|_\infty = 2^\nu + o(1)$. To make the $o(1)$ more precise, we need two results on Toeplitz matrices. The following proposition is certainly known because it delivers the spectral norm of Jordan blocks, whereas the following theorem seems to be less known. We have not found these two results in the literature and therefore will cite them with full proofs. Of course, these two results are also of interest by themselves.

Proposition 5.1. *Let $N \geqslant 2, \alpha > 0$, and consider the $N \times N$ matrix*

$$J_N := T_N(1+\alpha e^{-ix}) = \begin{pmatrix} 1 & \alpha & & & \\ & 1 & \alpha & & \\ & & \ddots & \ddots & \\ & & & 1 & \alpha \\ & & & & 1 \end{pmatrix}.$$

Then

$$\|J_N\|_\infty^2 = 1 + \alpha^2 + 2\alpha\cos\theta_0$$

where θ_0 is the smallest root of the equation $\sin(N+1)\theta + \alpha\sin N\theta = 0$ on the interval $(0,\pi)$.

Proof. We have

$$J_N^* J_N = \begin{pmatrix} 1 & \alpha & & & \\ \alpha & 1+\alpha^2 & \alpha & & \\ & \alpha & 1+\alpha^2 & \alpha & \\ & & \cdots & \cdots & \\ & & \alpha & 1+\alpha^2 & \end{pmatrix} = (1+\alpha^2)I + \alpha(\Delta_N - \alpha E_{11})$$

where $E_{11} = \operatorname{diag}(1,0,\dots,0)$ and Δ_N is the $N \times N$ Toeplitz matrix

$$\Delta_N = T_N(e^{ix} + e^{-ix}) = \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & & & & \\ & & & & & 1 \\ & & & & 1 & 0 \end{pmatrix}.$$

Thus, $\|J_N\|_\infty^2 = 1 + \alpha^2 + \alpha\lambda_{\max}$ where $\lambda_{\max}$ is the largest eigenvalue of $\Delta_N - \alpha E_{11}$. Obviously, $\|J_N\|_\infty^2 \leqslant (1+\alpha)^2 = 1 + \alpha^2 + 2\alpha$, which implies that $\lambda_{\max} \leqslant 2$. Clearly,

$$\det(\Delta_N - \alpha E_{11} - \lambda I) = (-\alpha - \lambda) D_{N-1}(\lambda) - D_{N-2}(\lambda) \tag{25}$$

with $D_k(\lambda) = \det(\Delta_k - \lambda I)$. Since $D_k(2) = (-1)^k(k+1)$, the right-hand side of (25) is never zero for $\lambda = 2$. Thus, $\lambda_{\max} < 2$. Possible eigenvalues of $\Delta_N - \alpha E_{11}$ in $(-2,2)$ are of the form $\lambda = 2\cos\theta$ with $\theta \in (0,\pi)$. It is well known that

$$D_k(2\cos\theta) = (-1)^k \frac{\sin(k+1)\theta}{\sin\theta}$$

(see, e.g., formula (2.10) of [2]). Consequently, (25) is zero if and only if

$$\begin{aligned} 0 &= (\alpha + 2\cos\theta)\sin N\theta - \sin(N-1)\theta \\ &= \alpha\sin N\theta + 2\cos\theta\sin N\theta - \sin N\theta\cos\theta + \cos N\theta\sin\theta \\ &= \alpha\sin N\theta + \cos\theta\sin N\theta + \cos N\theta\sin\theta \\ &= \alpha\sin N\theta + \sin(N+1)\theta. \end{aligned} \tag{26}$$

The value of (26) is positive for $\theta_1 = \pi/(N+1)$ and negative for $\theta_2 = 2\pi/(N+1)$. Consequently, (26) has a root in (θ_1, θ_2) and thus in $(0, \pi)$. It follows that $\lambda_{\max} = 2\cos\theta_0$ where θ_0 is the smallest root in $(0, \pi)$.

We are now in a position to prove (11). We know that $\beta_n^{(1)} = \|A_n^{-1}\|_\infty = \|T_n(c)\|_\infty$ with $c(x) = 1 + e^{-ix}$ and can hence use Proposition 5.1 with $\alpha = 1$. The smallest root of

$$0 = \sin(n+1)\theta + \sin n\theta = 2\sin\frac{(2n+1)\theta}{2}\cos\frac{\theta}{2}$$

in $(0, \pi)$ is $\theta_0 = 2\pi/(2n+1)$ and hence

$$\|T_n(c)\|_\infty^2 = 2 + 2\cos\frac{2\pi}{2n+1} = 4\cos^2\frac{\pi}{2n+1},$$

which completes the proof of (11). $\square$.

Theorem 5.2. *Let $a(x) = \sum_{k=-\nu}^{\nu} a_k e^{ikx}$ be a trigonometric polynomial with real coefficients a_k, suppose $\varphi(x) := |a(x)|^2$ reaches its maximum $\|a\|_\infty^2$ at $x = 0$ and nowhere else on $[-\pi, \pi)$, and assume $\varphi''(0) < 0$. Then*

$$\|T_N(a)\|_\infty = \|a\|_\infty\left(1 - \frac{|\varphi''(0)|\pi^2}{4\|a\|_\infty^2 N^2}\right) + O\left(\frac{1}{N^3}\right).$$

Proof. There is an identity for the product of two finite Toeplitz matrices which was first written down by Widom [28] and can also be found in [2, Proposition 3.10] and [3, formula (2.13)]. In our case it yields that

$$T_N^*(a)T_N(a) = T_N(|a|^2) - \begin{pmatrix} X_\nu & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & Y_\nu \end{pmatrix}, \tag{27}$$

where $T_N^*(a) = T_N(\overline{a})$ is the adjoint (=transpose) of $T_N(a)$ and X_ν and Y_ν are the $\nu \times \nu$ matrices

$$X_\nu = \begin{pmatrix} a_{-1} & \dots & a_{-\nu} \\ \vdots & \iddots & \\ a_{-\nu} & & \end{pmatrix}\begin{pmatrix} a_{-1} & \dots & a_{-\nu} \\ \vdots & \iddots & \\ a_{-\nu} & & \end{pmatrix},$$

$$Y_\nu = \begin{pmatrix} a_1 & & \\ \vdots & \ddots & \\ a_\nu & \dots & a_\nu \end{pmatrix}\begin{pmatrix} a_1 & & \\ \vdots & \ddots & \\ a_\nu & \dots & a_\nu \end{pmatrix}.$$

The matrices X_ν and Y_ν are obviously positive semidefinite. Therefore

$$\|T_N^*(a)T_N(a)\|_\infty \leqslant \|T_N(|a|^2)\|_\infty. \tag{28}$$

Deleting a row or a column of a matrix does not increase its norm. Due to (27), the matrix $T_{N-2\nu}(|a|^2)$ results from $T_N^*(a)T_N(a)$ by deleting the first and last ν rows and the first and last ν columns. Consequently,

$$\|T_N^*(a)T_N(a)\|_\infty \geqslant \|T_{N-2\nu}(|a|^2)\|_\infty. \tag{29}$$

The function $|a|^2$ is even, $|a(-x)|^2 = |a(x)|^2$, because the coefficients a_k are real. We can therefore have recourse to a theorem by Kac et al. [13] (or also to Theorem 2.1 of [25]), which gives that

$$\|T_m(|a|^2)\|_\infty = \|a\|_\infty^2 - \frac{|\varphi''(0)|\pi^2}{2m^2} + O\left(\frac{1}{m^3}\right)$$

as $m \to \infty$. Inserting this formula with $m = N$ in (28) and with $m = N - 2\nu$ in (29), we get

$$\|T_N^*(a)T_N(a)\|_\infty = \|a\|_\infty^2 - \frac{|\varphi''(0)|\pi^2}{2N^2} + O\left(\frac{1}{N^3}\right),$$

and taking the square root we arrive at the assertion. □

We now use Theorem 5.2 with $a(x)$ replaced by $c(x) = (1 + e^{-ix})^\nu$. We have $\|c\|_\infty = 2^\nu$, the function

$$\varphi(x) = |1 + e^{-ix}|^{2\nu} = 2^{2\nu}\left(\cos\frac{x}{2}\right)^{2\nu}$$

attains its maximum in $[-\pi, \pi)$ only at $x = 0$, and $\varphi''(0) = -2^{2\nu}\nu/2$. Thus, Theorem 5.2 yields that

$$\|T_N(c)\|_\infty = 2^\nu\left(1 - \frac{\pi^2}{4N^2}\frac{2^{2\nu}\nu}{2}\frac{1}{2^{2\nu}}\right) + O\left(\frac{1}{N^3}\right),$$

which immediately leads to (12).

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References

[1] J.A. Adell, E.A. Gallardo-Gutiérrez, The norm of the Riemann–Liouville operator on $L^p[0,1]$: a probabilistic approach, Bull. London Math. Soc. 39 (2007) 565–574.
[2] A. Böttcher, S. Grudsky, Spectral Properties of Banded Toeplitz Matrices, SIAM, Philadelphia, 2005.
[3] A. Böttcher, B. Silbermann, Introduction to Large Truncated Toeplitz Matrices, Springer-Verlag, New York, 1999.
[4] A. Böttcher, J. Virtanen, Norms of Toeplitz matrices with Fisher–Hartwig symbols, SIAM J. Matrix Anal. Appl. 29 (2007) 660–671.
[5] A. Böttcher, H. Widom, On the eigenvalues of certain canonical higher-order ordinary differential operators, J. Math. Anal. Appl. 322 (2006) 990–1000.
[6] A. Böttcher, H. Widom, From Toeplitz eigenvalues through Green's kernels to higher-order Wirtinger–Sobolev inequalities, Oper. Theory Adv. Appl. 171 (2007) 73–87.
[7] P. Dörfler, New inequalities of Markov type, SIAM J. Math. Anal. 18 (1987) 490–494.
[8] P. Dörfler, A Markov type inequality for higher derivatives of polynomials, Monatsh. Math. 109 (1990) 113–122.
[9] P. Dörfler, An estimation for the maximum singular value of a matrix, Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II 202 (1993) 187–194.
[10] S.P. Eveson, Norms of iterates of Volterra operators on L^2, J. Operator Theory 50 (2003) 369–386.
[11] Z.M. Franco, H.G. Kaper, Man Kam Kwong, A. Zettl, Best constants in norm inequalities for derivatives on a half-line, Proc. Roy. Soc. Edinburgh Sect. A 100 (1985) 67–84.
[12] P. Halmos, A Hilbert Space Problem Book, D. van Nostrand, Princeton, 1967.
[13] M. Kac, W.L. Murdock, G. Szegö, On the eigenvalues of certain Hermitian forms, J. Rational Mech. Anal. 2 (1953) 767–800.
[14] D. Kershaw, Operator norms of powers of the Volterra operator, J. Integral Equations Appl. 11 (1999) 351–362.
[15] N. Lao, R. Whitley, Norms of powers of the Volterra operator, Integral Equations Operator Theory 27 (1997) 419–425.
[16] G. Little, J.B. Reade, Estimates for the norm of the nth indefinite integral, Bull. London Math. Soc. 30 (1998) 539–542.
[17] W.A. Markoff, Über die Funktionen, die in einem gegebenen Intervall möglichst wenig von Null abweichen, Math. Ann. 77 (1916) 213–258.
[18] D.S. Mitrinović, J.E. Pečarić, A.M. Fink, Inequalities Involving Functions and Their Integrals and Derivatives, Kluwer, Dordrecht, 1991.
[19] E. Schmidt, Über die nebst ihren Ableitungen orthogonalen Polynomensysteme und das zugehörige Extremum, Math. Ann. 119 (1944) 165–204.
[20] L.F. Shampine, Some L_2 Markoff inequalities, J. Res. Nat. Bur. Standards 69B (1965) 155–158.
[21] P. Tarazaga, Eigenvalue estimates for symmetric matrices, Linear Algebra Appl. 135 (1990) 171–179.
[22] P. Tarazaga, More estimates for eigenvalues and singular values, Linear Algebra Appl. 149 (1991) 97–110.
[23] B. Thorpe, The norm of powers of the indefinite integral operator on (0, 1), Bull. London Math. Soc. 30 (1998) 543–548.
[24] P. Turán, Remark on a theorem of Erhard Schmidt, Mathematica (Cluj) 2 (25) (1960) 373–378.
[25] H. Widom, On the eigenvalues of certain Hermitian operators, Trans. Amer. Math. Soc. 88 (1958) 491–522.
[26] H. Widom, Extreme eigenvalues of translation kernels, Trans. Amer. Math. Soc. 100 (1961) 252–262.
[27] H. Widom, Extreme eigenvalues of N-dimensional convolution operators, Trans. Amer. Math. Soc. 106 (1963) 391–414.
[28] H. Widom, Asymptotic behavior of block Toeplitz matrices and determinants, II, Adv. Math. 21 (1976) 1–29.